INTERNATIONAL SYMPOSIUM on MUSICAL and ROOM ACOUSTICS

ISMRA 2016

September 11-13, 2016

La Plata, Buenos Aires, Argentina

Sound production - Sound synthesis: Paper ISMRA2016-82

Development of a modal Udwadia-Kalaba formulation for guitar modelling

Antunes Jose[a,b], **Debut Vincent**[a,b]

[a]Centro de Ciências e Tecnologias Nucleares, Instituto Superior Técnico, Universidade de Lisboa, Estrada Nacional 10, Km 139.7, 2695-066 Bobadela LRS, Portugal. jantunes@ctn.tecnico.ulisboa.pt
[b]Instituto de Etnomusicologia - Centro de Estudos em Música e Dança, Faculdade de Ciências Sociais e Humanas, Universidade Nova de Lisboa, Av. de Berna 26C, 1069-061 Lisbon, Portugal. vincentdebut@fcsh.unl.pt

Abstract

Most musical instruments consist on a set of dynamical subsystems connected at a number of constraining points through which energy flows or tuning is achieved. For any physical sound synthesis, one important difficulty deals with the manner to enforce these coupling constraints. While standard techniques include the use of Lagrange multipliers or penalty methods, we explore in this paper a different approach, the Udwadia-Kalaba (U-K) formulation, which is rooted on analytical dynamics but avoids the use of Lagrange multipliers. Up to now, this general and very elegant formulation has been nearly exclusively used for conceptual systems of discrete masses or articulated rigid bodies, namely in robotics. Despite its natural extension to deal with flexible systems modelled through their unconstrained modes, such an approach is surprisingly absent from the literature. Here, we show the potential of combining the U-K equations for constrained systems with the modal description, in order to simulate musical instruments. Our objectives are twofold: (1) to develop the U-K equations for constrained flexible systems in which subsystems are modelled through unconstrained modes, and (2) to apply this framework to compute the coupled dynamics of the string/body vibration. This example complements our work on guitar string modelling using penalty methods, and enables to compare results obtained from different approaches. Simulations show that the proposed technique provides results of comparable quality with a significant improvement in computational efficiency.

Keywords: String/body coupling, constrained dynamics, modal approach, time-domain simulations, guitar

INTERNATIONAL SYMPOSIUM on MUSICAL and ROOM ACOUSTICS
ISMRA 2016
September 11-13, 2016
La Plata, Buenos Aires, Argentina

Development of a modal Udwadia-Kalaba formulation for guitar modelling

1 Introduction

Most musical instruments consist on a set of dynamical subsystems connected at a number of constraining points, through which the vibratory energy flows or tuning can be achieved. Coupling is therefore an essential feature in instrument modelling and, when addressing physically-based synthesis, most modelling and computational difficulties are connected with the manner in which the coupling constraints are enforced. Typically, these are modelled using standard techniques such as Lagrange multipliers or penalty methods, each one with specific merits and drawbacks. In this paper we explore a different approach, the Udwadia-Kalaba (U-K) formulation, originally proposed in the early 90s for discrete constrained systems [1, 2], which is anchored on analytical dynamics but avoids the use of Lagrange multipliers.

Up to now, this general, very elegant and appealing formulation has been nearly exclusively used to address conceptual systems of discrete masses or articulated rigid bodies, namely in robotics. To the authors best knowledge, the single exception in the literature is the work by Pennestri et al. [3], who addressed a flexible slider-crank mechanism modelled using a Finite Element Timoshenko beam formulation. However, in spite of the possible natural extension of the U-K formulation to deal with flexible systems modelled through their unconstrained modes, such promising approach is surprisingly absent from the literature. In the present work we develop the potential of combining the U-K formulation for constrained systems with the modal description of flexible structures, in order to achieve reliable and efficient computations of dynamical responses, in particular for simulating the transient responses of musical instruments.

The objectives of this paper are thus twofold: (1) to develop the Udwadia-Kalaba equations for constrained flexible systems in which the various sub-structures are modelled through unconstrained modal basis, and (2) to apply this formulation to compute the dynamical responses of a guitar string coupled to the instrument body at the bridge. This illustration complements extensive work already performed in the past by the authors on guitar string/modelling using penalty methods, see Marques et al. [4] and Debut et al. [5], thus enabling an interesting comparison between the computational efficiency using different modelling techniques. These results demonstrate the computational efficiency of the proposed technique, which for the application at hand, achieved simulations of comparable quality, at least with an one-order of magnitude improvement in computational efficiency.

2 Theoretical formulation

Following on from Gauss' Principle of Least Action, Udwadia & Kalaba developed an elegant analytical treatment for the dynamics of discrete mechanical systems subjected to constraints. In general mathematical terms, they proposed the following standard form for the study of the

I S M R A 2 0 1 6
INTERNATIONAL SYMPOSIUM on MUSICAL and ROOM ACOUSTICS
September 11-13, 2016
La Plata, Buenos Aires, Argentina

constrained dynamics

$$\mathbf{M}\ddot{\mathbf{x}} = \mathbf{F}_e(t) + \mathbf{F}_c(t) \tag{1}$$

which simply expresses, according to Newton's second law of motion, the system response $\mathbf{x}$ as the result of the application of an applied force field $\mathbf{F}_e(t)$ and some additional forces $\mathbf{F}_c(t)$ stemming from a set of constraints. The U-K formulation then benefits from an alternative expression of the usual constraint equations $\psi_i(\mathbf{x}, \dot{\mathbf{x}}, t) = 0 \ (i = 1, \ldots, m)$, obtained by differentiation with respect to time, which leads to a set of linear equality relations in terms of the system accelerations, as:

$$\mathbf{A}(\mathbf{x}, \dot{\mathbf{x}}, t)\, \ddot{\mathbf{x}}(t) = \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t)\,, \tag{2}$$

where $\mathbf{A}$ is referred to as the *constraint matrix* and $\mathbf{b}$ is a known vector. Their main result is then to provide an explicit expression for the constrained dynamics $\mathbf{x}(t)$ and the constraint force vector $\mathbf{F}_c(t)$, at each instant, which are respectively given by the fundamental equation and the constraint force equation:

$$\ddot{\mathbf{x}} = \ddot{\mathbf{x}}_u + \mathbf{M}^{-1/2}\mathbf{B}^{+}(\mathbf{b} - \mathbf{A}\ddot{\mathbf{x}}_u) \tag{3a}$$

$$\mathbf{F}_c(t) = \mathbf{M}^{1/2}\mathbf{B}^{+}(\mathbf{b} - \mathbf{A}\ddot{\mathbf{x}}_u) \tag{3b}$$

denoting $\mathbf{B}^{+}$ the Moore-Penrose pseudo-inverse of the matrix $\mathbf{B} = \mathbf{A}\mathbf{M}^{-1/2}$ which has the property of uniqueness. In Eqs. (3), the vector $\ddot{\mathbf{x}}_u$ represents the dynamical response of the unconstrainted system, i.e. when no constraint is imposed, which is solution of:

$$\ddot{\mathbf{x}}_u = \mathbf{M}^{-1}\mathbf{F}_e(t) \tag{4}$$

while the second term in the right-side hand of Eq. (3a) accounts for the influence of the constraints on the system dynamics, since the unconstrainted system must further comply with the physical constraints. The superlative elegance of the U-K formulation lays in the fact that it encapsulates, in a single explicit equation (3a), both the dynamical equations of the system and the constraints applied, and allows, if needed, the knowledge of the constraining force through (3b). In particular, no additional variables, such as Lagrange multipliers, are needed. Notably, Eqs. (3) may be applied to linear or nonlinear, conservative or dissipative systems, and may also be efficiently solved using a suitable time-step integration scheme for a given excitation. Finally, it can be noticed that if no constraints are applied, the constraint force vector, and thus the correcting term in (3a) is nil, so that the usual unconstrained formulation is recovered.

3 Modal formulation for the U-K equations

3.1 Fundamental equation

Although originally proposed for constrained discrete systems, the Udwadia-Kalaba formulation can be extended to deal with continuous flexible systems contrained at some specific points. Adopting a modal framework in which any physical quantity $\mathbf{x}(\mathbf{r}, t)$ varying in space and time is expressed as a modal superposition,

$$\mathbf{x}(\mathbf{r}, t) = \sum_{n=1}^{N} q_n(t)\phi_n(x)\,, \tag{5}$$

where $q_n(t)$ are the modal amplitudes and $\phi_n(x)$ the modeshapes appropriate to the boundary conditions, a formulation very similar to the previous U-K equation (3a) can be derived for the modal responses of the constrained system. Projecting as usual the dynamical equations on the modeshapes of the system, this yields:

$$\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{q}}_u + \boldsymbol{M}^{-1/2}\boldsymbol{B}^{+}(\mathbf{b} - \boldsymbol{A}\ddot{\boldsymbol{q}}_u) \tag{6}$$

where $\ddot{\boldsymbol{q}}_u$ are the modal accelerations of the unconstrained configuration, $\boldsymbol{M} = \Phi^T \mathbf{M} \Phi$ is the modal mass matrix $\boldsymbol{A} = \mathbf{A}\Phi_c$ is the modal constraint matrices, $\boldsymbol{B} = \boldsymbol{A}\boldsymbol{M}^{-1/2}$, Φ being the modal matrices and Φ_c the modal matrix of all modeshapes where constraints are defined. From the knowledge of the constrained modal responses, the modal constraining force can also be computed by multiplying the correcting term of Eq. (6) by the system modal mass matrix $\boldsymbol{M}$. Finally, any modal quantities can then be converted into the corresponding physical quantities by modal summation of the standard form (5).

3.2 Coupled dynamical subsystems

The application of the U-K modal formulation to a set of vibrating subsystems coupled through a number of kinematic constraints, can now be considered. In a modal description, the dynamics of a given subsystem s subjected to an external force field, can be classically written as a set of modal equations in terms of the modal amplitudes $\boldsymbol{q}^s$ and its derivatives,

$$\boldsymbol{M}^s\ddot{\boldsymbol{q}}^s + \boldsymbol{C}^s\dot{\boldsymbol{q}}^s + \boldsymbol{K}^s\boldsymbol{q}^s + \boldsymbol{F}^s_{nl}(\boldsymbol{q}^s, \dot{\boldsymbol{q}}^s) = \boldsymbol{F}^s_{ext} \qquad s = 1, 2, \ldots, S \tag{7}$$

where $\boldsymbol{M}^s$, $\boldsymbol{C}^s$ and $\boldsymbol{K}^s$ are diagonal matrices of the modal parameters, and $\boldsymbol{F}^s_{ext}$ and $\boldsymbol{F}^s_{nl}$ are the modal force vectors stemming from the external and nonlinear force fields, which are obtained by projection of the physical forces on the modal basis. According to Eq. (6), the U-K approach requires the computation of the modal accelerations of the unconstrained system, which are simply given by:

$$\ddot{\boldsymbol{q}}^s_u = \left(\boldsymbol{M}^s\right)^{-1}\boldsymbol{F}^s \qquad s = 1, 2, \ldots, S \tag{8}$$

where $\boldsymbol{F}^s$ denotes the vector of all the constraint-independent modal forces:

$$\boldsymbol{F}^s = \boldsymbol{F}^s_{ext} - \boldsymbol{C}^s\dot{\boldsymbol{q}}^s - \boldsymbol{K}^s\boldsymbol{q}^s - \boldsymbol{F}^s_{nl}(\boldsymbol{q}^s, \dot{\boldsymbol{q}}^s), \qquad s = 1, 2, \ldots, S \tag{9}$$

for which it is assumed that the vectors of modal constrained displacements and velocities are known at each time-step. Assembling the modal quantities of the S subsystems in compact vectors and block matrices, the unconstrained modal accelerations of the coupled system $\boldsymbol{Q}_u = [\boldsymbol{q}^1_u, \boldsymbol{q}^2_u, \ldots, \boldsymbol{q}^S_u]^T$ read finally as:

$$\ddot{\boldsymbol{Q}}_u = \mathsf{M}^{-1}\left[\mathsf{F}_{ext} - \mathsf{C}\dot{\boldsymbol{Q}} - \mathsf{K}\boldsymbol{Q} - \mathsf{F}_{nl}(\boldsymbol{Q}, \dot{\boldsymbol{Q}})\right] \tag{10}$$

where the matrices M, C and K are block diagonal matrices set up by the submatrices of the modal parameters of the various subsystems, built according to:

$$\mathsf{M} \equiv \begin{bmatrix} \boldsymbol{M}^1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{M}^2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \boldsymbol{M}^S \end{bmatrix} \quad \mathsf{C} \equiv \begin{bmatrix} \boldsymbol{C}^1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{C}^2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \boldsymbol{C}^S \end{bmatrix} \quad \mathsf{K} \equiv \begin{bmatrix} \boldsymbol{K}^1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{K}^2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \boldsymbol{K}^S \end{bmatrix} \tag{11}$$

where the modal parameters of the matrices $\boldsymbol{M}^s$, $\boldsymbol{C}^s$ and $\boldsymbol{K}^s$, defined according to the modeshapes ϕ_n^s, are given respectively as:

$$m_n^s = \int_{D_s} \rho(\mathbf{r}^s)\left[\phi_n^s(\mathbf{r}^s)\right]^2 d\mathbf{r}^s, \quad c_n^s = 2 m_n^s \omega_n^s \zeta_n^s, \quad \text{and} \quad k_n^s = m_n^s (\omega_n^s)^2 \tag{12}$$

denoting ω_n^s the circular eigenfrequency and ζ_n^s the damping value. In Eq. (10), the column vector $\boldsymbol{Q} = [\boldsymbol{q}^1, \boldsymbol{q}^2, \ldots, \boldsymbol{q}^S]^T$ and $\dot{\boldsymbol{Q}}$ describes the modal displacements and velocities for the S constrained subsystems, while F_{ext} and F_{nl} are the modal vectors associated to the external and nonlinear interaction forces respectively.

Besides the unconstrained equation (10), the second set of equations to be considered in the U-K formulation concerns the P constraints which couple the various subsystems, defined at specific locations $\mathbf{r}_c^s$. In most practical situations, these are amenable to linear relationships by appropriate differentiation with respect to time, leading to the standard form:

$$\boldsymbol{A}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t)\ddot{\boldsymbol{Q}} = \mathbf{b}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t) \tag{13}$$

where $\boldsymbol{A} = \mathbf{A}\Phi_c$ with

$$\Phi_c = \begin{bmatrix} \Phi_c^1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \Phi_c^2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \Phi_c^S \end{bmatrix} \tag{14}$$

where the Φ_c^s contains the modeshape vectors of each subsystem at the constraint location $\mathbf{r}_c^s$. Finally, from Eqs. (10) and (13), one can compute at each time-step the constrained modal accelerations through Eq. (6), as well as the constraining forces if needed.

4 The coupled guitar string/body/player vibration

For illustrative purpose, we formally address the coupled dynamics of a guitar string and body, coupled at the instrument bridge, and including the influence of a stopping finger along the fingerboard to control the frequency of the plucked tone. The vibrating elements are modelled using the modal U-K formulation, using the unconstrained modal basis of the string and of the instrument body, while the string/finger coupling is thought as a simple physical constraint acting on the vibratory motion of the string.

4.1 Solutions for the unconstrained modal equations

According to (7), the forced response of the string can be formulated as a set of N_S secondary-order ODEs, written in the matrix form as:

$$\boldsymbol{M}^S \ddot{\boldsymbol{q}}^S(t) + \boldsymbol{C}^S \dot{\boldsymbol{q}}^S(t) + \boldsymbol{K}^S \boldsymbol{q}^S(t) = \boldsymbol{f}_{ext}(t) \tag{15}$$

where the vector $\boldsymbol{q}^S(t)$ describes the modal amplitudes of the string and $\boldsymbol{f}_{ext}(t)$ is the modal force corresponding to the pluck excitation. Similarly, the modal response of the body can be formulated as a set of N_B secondary-order ODEs, written in the matrix form as:

$$\boldsymbol{M}^B \ddot{\boldsymbol{q}}^B(t) + \boldsymbol{C}^B \dot{\boldsymbol{q}}^B(t) + \boldsymbol{K}^B \boldsymbol{q}^B(t) = \mathbf{0} \tag{16}$$

where $\boldsymbol{q}^B(t)$ are the modal amplitudes of the body motion. All the matrices in Eqs. (15) and (16) are diagonal matrices of the modal parameters of the string and the instrument body respectively, defined through their unconstrained modal basis. Compared to penalty formulation, it is important to note that no external body forces are considered in Eqs. (15) and (16) as the string/body coupling is formulated in terms of a constraint equation expressed at the location of the bridge. Grouping together the dynamical equations (15) and (16), the modal responses of the unconstrained system are formulated as a set of $N_S + N_B$ modal equations, written as:

$$\begin{Bmatrix} \ddot{\boldsymbol{q}}_u^S \\ \ddot{\boldsymbol{q}}_u^B \end{Bmatrix} = \begin{bmatrix} (\boldsymbol{M}^S)^{-1} & \mathbf{0} \\ \mathbf{0} & (\boldsymbol{M}^B)^{-1} \end{bmatrix} \left(- \begin{bmatrix} \boldsymbol{C}^S & \mathbf{0} \\ \mathbf{0} & \boldsymbol{C}^B \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{q}}^S \\ \dot{\boldsymbol{q}}^B \end{Bmatrix} - \begin{bmatrix} \boldsymbol{K}^S & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^B \end{bmatrix} \begin{Bmatrix} \boldsymbol{q}^S \\ \boldsymbol{q}^B \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{f}_{ext}(t) \\ \mathbf{0} \end{Bmatrix} \right) \tag{17}$$

The matrices in Eq. (17) are block diagonal matrices, which highlights that the unconstrained dynamics is treated as if the modal amplitudes were independent.

4.2 Modal constraints equations

String/body coupling. Assuming a rigid transmission at the bridge between the string and the body, the modal equations of the constrained system are required to satisfy a set of constraint modal equations, stemming from the condition that, at the bridge location, the string motion $Y^S(x_B,t)$ must be the same as the instrument body motion at the string location $Y^B(\mathbf{r_s},t)$. Formally, this results in the condition:

$$Y^S(x_b,t) - Y^B(\mathbf{r_s},t) = 0 \quad \Longleftrightarrow \quad \left[\Phi^S(x_B)\right]^T \boldsymbol{q}^S(t) - \left[\Phi^B(\mathbf{r_s})\right]^T \boldsymbol{q}^B(t) = \mathbf{0} \tag{18}$$

with the modeshape vectors

$$\Phi^S(x_B) = \left[\phi_1^S(x_B)\,\phi_2^S(x_B)\ldots\phi_{N_S}^S(x_B)\right]^T, \qquad \Phi^B(\mathbf{r_s}) = \left[\phi_1^B(\mathbf{r_s})\,\phi_2^B(\mathbf{r_s})\ldots\phi_{N_B}^B(\mathbf{r_s})\right]^T \tag{19}$$

String/finger coupling. Following a simple strategy for modelling the influence of the string/finger interaction, the string motion $Y^S(x_f,t)$ must be nil at the finger location x_f. To refine the modelling and account for the finite width of the finger, several rigids constraint can be thought at close locations x_{f_i}, so that the string/finger constraint equation becomes a set of F modal equations of the form:

$$Y^S(x_{f_i}(t),t) = 0 \quad \Longleftrightarrow \quad \left[\Phi^S(x_{f_i})\right]^T \boldsymbol{q}^S(t) = \mathbf{0} \qquad i = 1,\ldots,F \tag{20}$$

with the modeshape vector

$$\Phi^S(x_{f_i}) = \left[\phi_1^S(x_{f_i})\,\phi_2^S(x_{f_i})\ldots\phi_{N_S}^S(x_{f_i})\right]^T \tag{21}$$

Finally, assembling the two constraint conditions (18) and (20) into a compact form, the global constraint equation (13) reads as:

$$\boldsymbol{A}\ddot{\boldsymbol{Q}} = \mathbf{b} \quad \Longleftrightarrow \quad \begin{bmatrix} \left[\Phi^S(x_B)\right]^T & -\left[\Phi^B(\mathbf{r_s})\right]^T \\ \left[\Phi^S(x_{f_1})\right]^T & \mathbf{0} \\ \vdots & \vdots \\ \left[\Phi^S(x_{f_F})\right]^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{q}_1^S(t) \\ \vdots \\ \ddot{q}_{N_S}^S(t) \\ \ddot{q}_1^B(t) \\ \vdots \\ \ddot{q}_{N_B}^B(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix} \tag{22}$$

4.3 Solutions for the constrained modal equations

Finally, for the case of interest, the constrained modal accelerations are readily computed by the efficient recurrence:

$$\begin{Bmatrix} \ddot{\boldsymbol{q}}^S \\ \ddot{\boldsymbol{q}}^B \end{Bmatrix} = \begin{Bmatrix} \ddot{\boldsymbol{q}}_u^S \\ \ddot{\boldsymbol{q}}_u^B \end{Bmatrix} - \boldsymbol{M}^{-1/2}\boldsymbol{B}^{+}\boldsymbol{A} \begin{Bmatrix} \ddot{\boldsymbol{q}}_u^S \\ \ddot{\boldsymbol{q}}_u^B \end{Bmatrix} = \boldsymbol{W} \begin{Bmatrix} \ddot{\boldsymbol{q}}_u^S \\ \ddot{\boldsymbol{q}}_u^B \end{Bmatrix} \quad \text{with} \quad \boldsymbol{W} = \boldsymbol{1} - \boldsymbol{M}^{-1/2}\boldsymbol{B}^{+}\boldsymbol{A} \tag{23}$$

from which the physical motions of the string and the body can be computed by modal summation as usual.

5 Illustrative computations

5.1 System parameters

The present computations pertain to a single string with parameter values experimentally identified in the work [6]. The string total length is $L = 0.65$ cm, with axial tensioning force $T = 73.9$ N and mass per unit length 3.6111^{-3} Kg/m, transverse wave propagation velocity $c_T = 143$ m/s and bending stiffness $B = EI = 4.10^{-5}$ N.m^2, resulting in a fundamental frequency of 110 Hz. To be coherent with the U-K formulation, the string modal basis is that of the unconstained string, assumed pinned at the nut and *free* at the bridge, for which the modeshapes ϕ_n^S and modal frequencies f_n^S are given by:

$$\phi_n^S(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right), \quad \text{and} \quad f_n^S(x) = \frac{c_T}{2\pi} p_n \left(1 + \frac{B}{2T} p_n^2\right) \tag{24}$$

with $p_n = (2n-1)\pi/2L$ and $n = 1, \ldots, N_S$. Concerning the string modal damping, complex dissipative phenomena must be accounted for, as thoroughly discussed by Woodhouse [6], who proposed a pragmatic formulation for modal damping based on three loss parameters as:

$$\zeta_n^S = \frac{1}{2} \frac{T\left(\eta_F + \eta_A/\omega_n^S\right) + \eta_B B p_n^2}{T + B p_n^2} \tag{25}$$

where the loss coefficients related to *internal friction*, *air viscous damping* and *bending damping*, fitted from experimental data, are for this string $\eta_F = 7\,10^{-5}$, $\eta_A = 0.9$ and $\eta_B = 2.5\,10^{-5}$. A total number of 150 modes is considered in the computations, covering the frequency range up to 20000 Hz, which proves a sensible compromise for convergence. For illustration, the first modal parameters for the computed string are presented in Figure 1.

For the guitar body, the modal parameters were experimentally identified from a transfer function measured at the bridge of a real-life instrument, with impact excitation applied in the perpendicular direction of the soundboard. The input force was measured using a miniature force sensor (Kistler type 9211) while the vibratory reponse was sensed by an accelerometer (B&K 4375), both sensors being glued at the same location, on the bridge, close to the string/bridge interaction point. Modal identification was then achieved in the frequency domain by developing a MDOF algorithm based upon a curve fitting procedure - see [7]. The modal parameters experimentally identified in the range 0-800 Hz are presented in Figure 2.

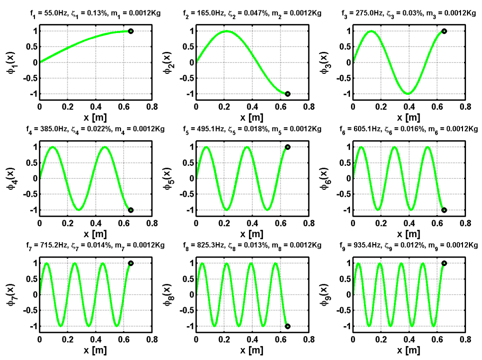

Figure 1: First modal parameters for the computed string.

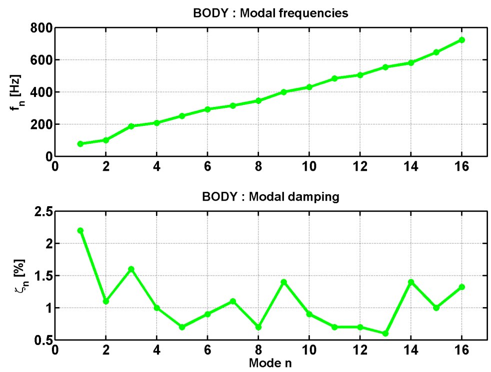

Figure 2: Modal frequencies and damping values of the instrument body.

5.2 Simulation results

In the presented simulations, the string is assumed to be plucked by a finger, at location x_e = 0.9L from the bridge, and to be stopped with a tuning constraint on the fingerboard at $x = 0.33L$, using a set of three rigid kinematic constraints. The string excitation is a linear force ramp 0-5 N, applied for the initial 10 ms of the simulation while the total simulation length is 10-s long. For the time-step integration, a simple explicit Velocity-Verlet algorithm was implemented - see for instance [8], using a time-step of 10^{-5} sec.

Figures 3 and 4 show the time-responses and corresponding spectra of the string motion, computed at the locations of the excitation, the bridge and the finger, for a string constrained by a stopping finger and a rigid motionless bridge, thus ignoring the vibrational behaviour of the guitar body. As one would expect, the string motion shows the classical exponential decay during the simulation, while it is virtually nil at the two constraint locations, thereby complying with the physical constraints. One can also notice the effect of the constraining finger on the string vibration in Figure 4, as the playing frequency is higher than both the fundamental frequency of the string (55 Hz) and the tuning frequency of the pinned-pinned string (110 Hz). Also, since the modeled string is not ideal, high frequency fluctuations can be seen in the string motion, prior to each reflexion pulse, which actually corresponds to the arrival of precursor flexural waves. As evidenced by the small-amplitude peaks visible in the spectra of Figure 4, a small amount of the vibratory energy actually flows from the excited region to the nominally passive region of the string, i.e from the finger to the nut. The reason can be found in the manner on how the string/finger constraint has been modeled in the present computation, directly formulated in terms of kinematic constraints and without any dissipative element. More complex modeling strategy can be thought for the string/finger interaction, in particular using flexible-dissipative-inertial elementary subsystems, which then implies a larger system size.

Figures 5 and 6 then pertain to computations including the multi-modal dynamics of the instrument body, which is coupled to the string at the bridge. A comparison between Figures 3 and 5 clearly highlights the influence of the mechanical response of the body on the string

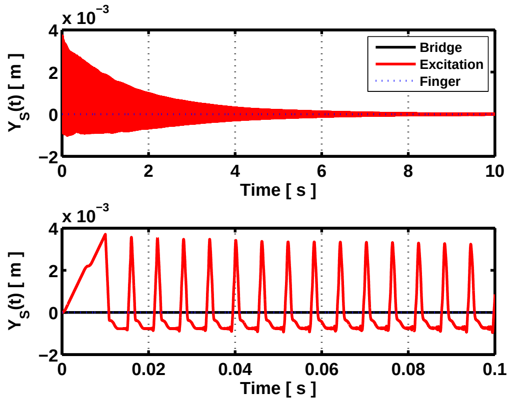

Figure 3: Time-history of string motion at different locations along the string. Rigid body.

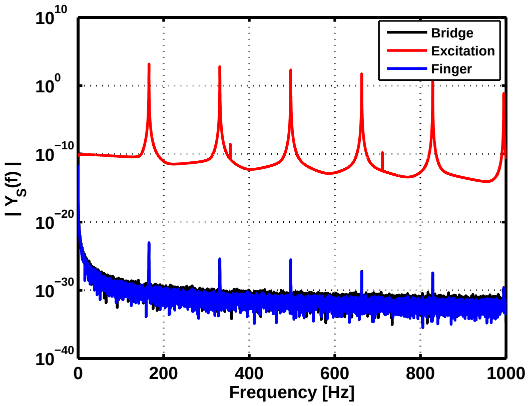

Figure 4: Spectra of the string motions at different locations along the string. Rigid body.

motion. In particular, the decrease of the string motion amplitude becomes more complex when the body dynamics is included, since vibrational energy is transferred back and forth between the two subsystems. Also, the global dissipative role of the instrument body is shown through the shorter live of the string response. On the other hand, one can also notice peaks in the response spectrum of the string which stem from the body modes. Finally, Figure 6 compares the motions of the body and the string computed at the bridge, in order to give an illustration of the kinematical constraint. As seen, the two curves are superimposed together, which is in perfect agreement with the kinematical constraint considered at the bridge - see Eq. (18).

6 Conclusions

In this paper we developed a new approach for computing the dynamics of coupled flexible systems, based on the general formulation of Udwadia-Kalaba which is becoming increasingly popular in the field of multibody dynamics. The general U-K equations were adapted to address coupled subsystems, linear or nonlinear, defined in terms of their unconstrained modal basis. The U-K formulation shows a considerable potential to deal effectively with the dynamics of physically modelled musical instruments, for which vibrational energy is exchanged between various subsystems and tuning is achieved at a number of constraining locations. Therefore, the formulation developed was applied on a guitar, including the fully coupled dynamics of a string, tuned somewhere on the fingerboard, and the instrument body. The illustrative computations presented highlight the role of the body dynamics on the string response, as well the influence of the stopping finger model, which can partially affects the vibratory responses of both the “active” and “passive” regions of the string. The results obtained are consistent with those previously obtained by the authors when modelling string/body coupled vibrations using penalty methods for the constraints. However, numerically, the present approach proved significantly more efficient, at least by an order of magnitude. The proposed approach is

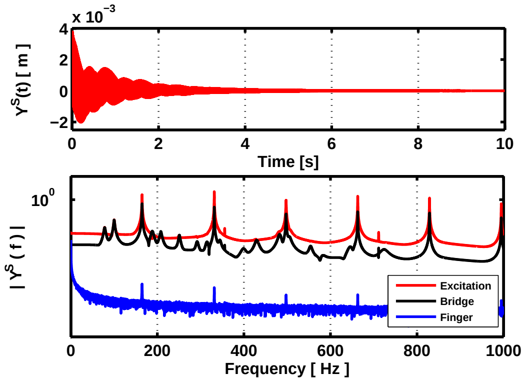

Figure 5: Time-history and spectra of string motion at 3 locations. Flexible body.

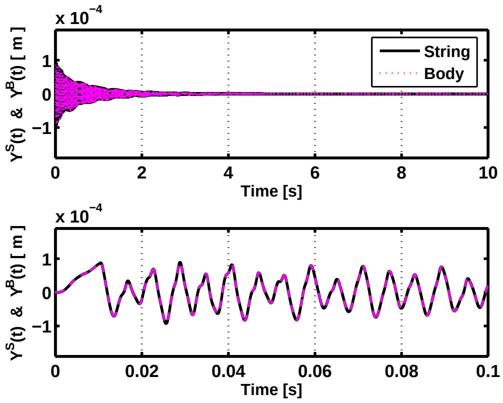

Figure 6: String and body motions at bridge location. Flexible body.

currently being extended to address intermittent constraints between subsystems.

Acknowledgements

The authors acknowledge the Fundação para Ciências e Tecnologia for the financial supports of C2TN and INET-md through the UID/EAT/00472/2013 and UID/EAT/00472/2013 projects.

References

[1] F.E. Udwadia and R.E. Kalaba. A new perspective on constrained motion. *Proceedings of the Royal Society*, pages 407–410, 1992.

[2] F.E. Udwadia and R.E. Kalaba. *Analytical dynamics - A new approach*. Cambridge University Press, 1996.

[3] E Pennestri, P.P. Valentini, and D. de Falco. An application of the udwadia-kalaba dynamic formulation to flexible multibody systems. *J. Franklin Inst. B*, 347:173–194, 2010.

[4] M. Marques, J. Antunes, and V. Debut. Coupled modes and time-domain simulations of a twelve-string guitar with a movable bridge. In *Proceedings of the Stockholm Music Acoustic Conference*, 2013.

[5] V. Debut, M. Carvalho, M. Marques, and J. Antunes. Physics-based modeling techniques of a twelve-string portuguese guitar: a three-dimensional non-linear time-domain computational approach for the multiple-strings/bridge/soundboard coupled dynamics. *Appl. Acoust.*, 108:3–18, 2016.

[6] J. Woodhouse. On the synthesis of guitar plucks. *Acta Acust. united Ac.*, 90:928–944, 2004.

[7] R.J. Allemang and D.L Brown. *Handbook on Experimental Mechanics*. Prentice Hall, 1994.

[8] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery. *Numerical Recipes 3rd Edition: The Art of Scientific Computing*. Cambridge University Press, New York, NY, USA, 3 edition, 2007.